Chapter 1 - Notation and Review of Newton's Laws

C. Newton's second law
D. Newton's third law and conservation of momentum

C. Newton's Second Law

Newton's Second Law

Newton's Second Law:

$$\mathbf{F} = m\mathbf{a}$$

Alternate Form:

$$\mathbf{F} = \dot{\mathbf{p}}$$

Where $\mathbf{F} =$ the net force on the object [N = kg m/s^2]

$m =$ the object's mass [kg]

$\mathbf{a} =$ the object's acceleration [m/s^2]

$\mathbf{p} =$ the object's momentum $(\mathbf{p} = m\mathbf{v})$ [kg m/s]

The "alternate form" is valid even when the mass changes with time while the "standard form" is generally not.

Newton's second law is valid in any **inertial frame** (a frame of reference that is not accelerating). In accelerating (noninertial) frames, additional "inertial forces" must be included (see chapter 9)

Newton's Second Law

Derivation of the alternate form of Newton's Second Law when mass is constant

We start with $\mathbf{F} = m\mathbf{a}$

Substitute $\mathbf{a} = d\mathbf{v}/dt$: $= m\dfrac{d}{dt}(\mathbf{v})$

Bring in mass to derivative $= \dfrac{d}{dt}(\mathbf{mv})$

$= \dot{\mathbf{p}}$ ✓

Newton's Second Law and ODEs

In one dimension Newton's second law leads to a second-order Ordinary Differential Equation (ODE):

$$m\ddot{x} = F_x$$

In two or three dimensions, Newton's second law in vector form $\mathbf{F} = m\mathbf{a}$ leads to a system of 2 or 3 of ODEs:

$$m\ddot{x} = F_x \qquad m\ddot{y} = F_y \qquad m\ddot{z} = F_z$$

In this course, you will gain experience solving a variety of second-order ODEs.

Example ODEs that arise from F=ma

One Dimension

Constant force $$m\ddot{x} = F_0$$

Spring force (S.H.O.) $$m\ddot{x} = -kx$$

Damped, driven oscillator $$m\ddot{x} = -kx - bv + F_0 \cos \omega_0 t$$

Two and Three Dimensions

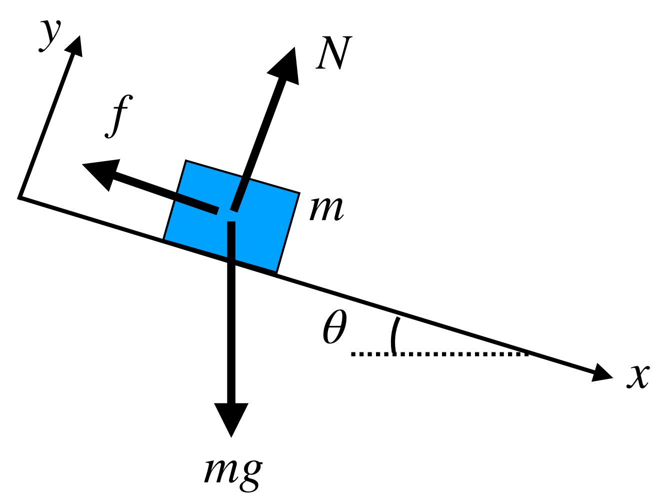
Projectile motion with quadratic drag $$m\ddot{\mathbf{r}} = -m\mathbf{g} - cv^2\hat{\mathbf{v}}$$

Charged particle in magnetic field $$m\dot{\mathbf{v}} = q\mathbf{v} \times \mathbf{B}$$

Example: Block sliding down an inclined plane

A block with mass m is sliding down a plane inclined by an angle θ w.r.t. the horizontal. The coefficient of friction is μ. Find the distance it slides in time t.

Solution.

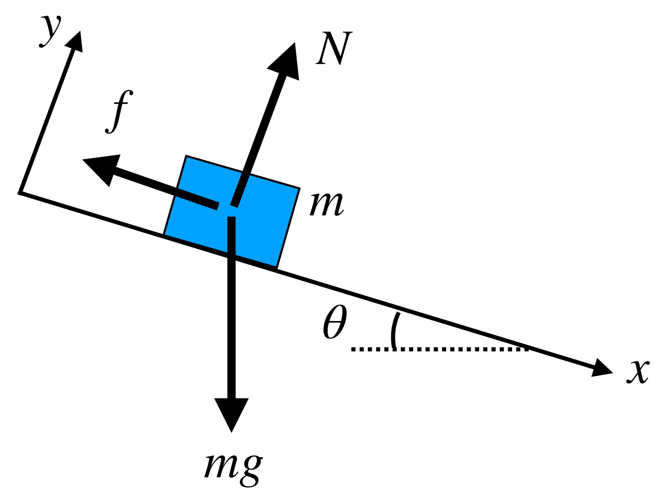


Step 1 - Draw a diagram and label all forces, parameters, etc.: $N =$ normal force perpendicular to plane, $f = \mu N =$ friction force parallel to plane, $mg =$ force of gravity in downward vertical direction

Step 2 - Pick a coordinate system that matches the problem. In this case we tip the x axis to be parallel to the inclined plane. Sketch the axes on the diagram.

Example: Block sliding down an inclined plane (cont.)

Step 3 - Apply Newton's second law to x and y axes independently.



x axis: The block will accelerate down the plane so there will be a nonzero acceleration:

$$ma = mg\sin\theta - f$$

y axis: The block will not leave the plane, so the forces in y direction will balance

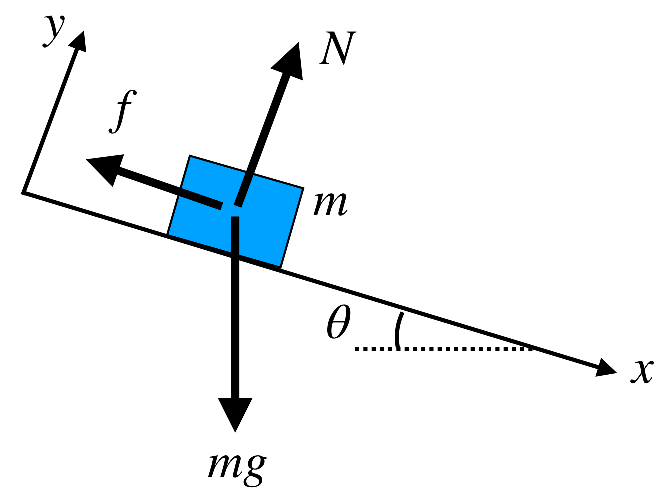
$$N = mg\cos\theta$$

Step 4 - Substitute in for $f = \mu N$ and solve for acceleration

x equation → $ma = mg\sin\theta - \mu N$

→ $a = g\sin\theta - \dfrac{\mu N}{m}$

Example: Block sliding down an inclined plane (cont.)



Step 5 - Substitute $N = mg\cos\theta$ in to acceleration equation and simplify

$$a = g\sin\theta - \frac{\mu N}{m}$$

$$= g\sin\theta - \frac{\mu mg\cos\theta}{m}$$

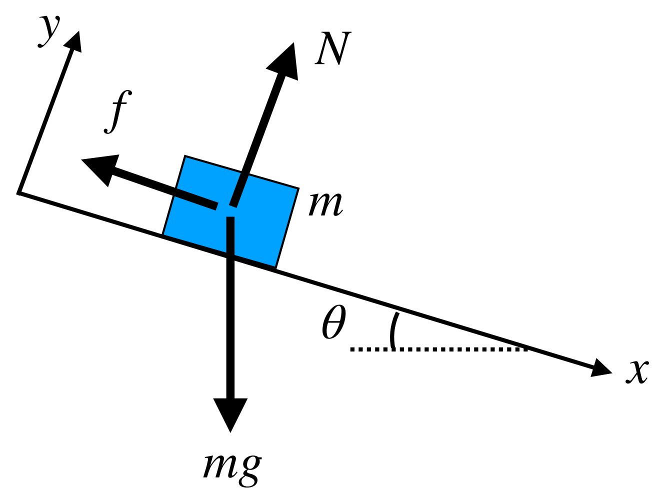
$$= g(\sin\theta - \mu\cos\theta)$$

Step 6 - Integrate twice to solve for $x(t)$. First write $a = dv/dt$.

$$\frac{dv}{dt} = g(\sin\theta - \mu\cos\theta) \quad \longrightarrow \quad \int_{v_0}^{v} dv = \int_0^t g(\sin\theta - \mu\cos\theta)dt$$

$$v(t) - v_0 = g(\sin\theta - \mu\cos\theta)t$$

Example: Block sliding down an inclined plane (cont.)



Step 6 (cont) - Write $v = dx/dt$ and integrate a second time.

$$\frac{dx}{dt} = v_0 + g(\sin\theta - \mu\cos\theta)t$$

$$\int_0^x dx = \int_{t_0}^t \big(v_0 + g(\sin\theta - \mu\cos\theta)t\big)\, dt$$

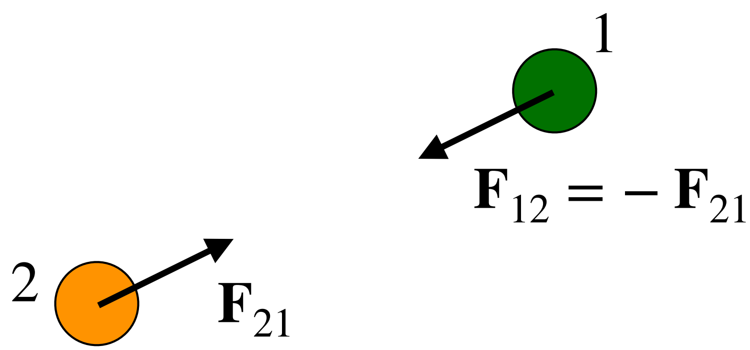
$$\boxed{x(t) = v_0 t + \frac{1}{2}g(\sin\theta - \mu\cos\theta)t^2}$$

Step 7 - Reality check. Evaluate units and limiting cases.

- Each term has units of length (meters) ✓
- When $\theta = 90^\circ$ and $\mu = 0$, the solution is $x(t) = v_0 t + \frac{1}{2}gt^2$ ✓
- When $\mu > \tan\theta$, the acceleration (t^2) term is negative meaning the block will slow to a stop. This makes sense qualitatively. ✓

D. Newton’s Third Law

Newton's Third Law



If object 1 exerts a force $\mathbf{F}_{21}$ on object 2, then object 2 exerts a reaction force $\mathbf{F}_{12}$ on object 1 given by

$$\mathbf{F}_{12} = -\mathbf{F}_{21}$$

Newton's 3rd law is valid in non-relatavistic classical mechanics calculations in the absence of time-dependent electromagnetic fields. (See Taylor, Chapter 1)

Newton's Third Law

Newton's 3rd law leads to conservation of momentum. (See Taylor, Chapter 1 for a discussion about the relation between Newton's third law and conservation of momentum).

Imagine two particles, 1 and 2, that each exert a force on each other and that feel an external force.

Net force on particle 1: $\mathbf{F}_1 = \mathbf{F}_{12} + \mathbf{F}_1^{ext}$

Net force on particle 2: $\mathbf{F}_2 = \mathbf{F}_{21} + \mathbf{F}_2^{ext}$

Apply Newton's 2nd law to each particle: $\dot{\mathbf{p}}_1 = \mathbf{F}_1$ and $\dot{\mathbf{p}}_2 = \mathbf{F}_2$.

Define the total momentum of the system to be $\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2$.

Take the derivative: $\dot{\mathbf{P}} = \dot{\mathbf{p}}_1 + \dot{\mathbf{p}}_2$ → $\dot{\mathbf{P}} = \mathbf{F}_1 + \mathbf{F}_2$

Newton's Third Law

Substitute the expression for the total force on each particle and rearrange:

$$\dot{\mathbf{P}} = \mathbf{F}_1 + \mathbf{F}_2 = \left(\mathbf{F}_{12} + \mathbf{F}_1^{ext}\right) + \left(\mathbf{F}_{21} + \mathbf{F}_2^{ext}\right)$$
$$= \left(\mathbf{F}_{12} + \mathbf{F}_{21}\right) + \left(\mathbf{F}_1^{ext} + \mathbf{F}_2^{ext}\right)$$

Define the net external force on system as $\mathbf{F}^{ext} = \mathbf{F}_1^{ext} + \mathbf{F}_2^{ext}$, so we have

$$\dot{\mathbf{P}} = \left(\mathbf{F}_{12} + \mathbf{F}_{21}\right) + \mathbf{F}^{ext}$$

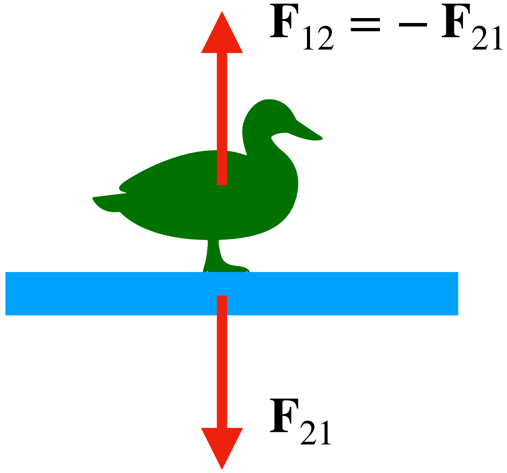
Newton's third law tells us $\mathbf{F}_{12} = -\mathbf{F}_{21}$, so the term in the parenthesis (which is the sum of the internal forces) is zero. This leaves:

$$\dot{\mathbf{P}} = \mathbf{F}^{ext}$$

This result tells us that momentum is conserved ($\dot{\mathbf{P}} = 0$) if the net external force is zero. ✓

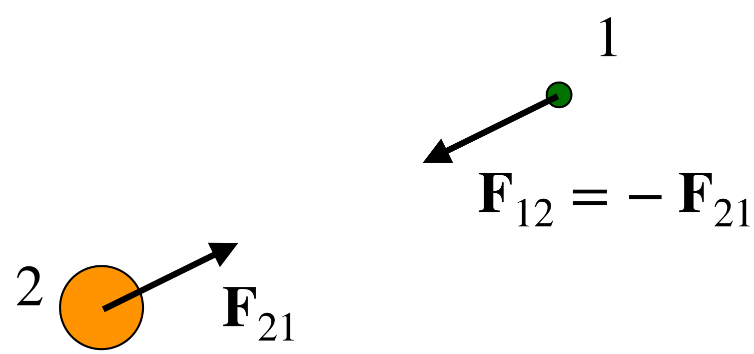
Examples of Newton's 3rd Law

Weight of a duck pressing down on the ground equals the normal force of the ground pushing up on the duck.



The forces are equal and opposite.

Electrostatic force of an electron on a proton is equal and opposite to the electrostatic force of a proton on the electron

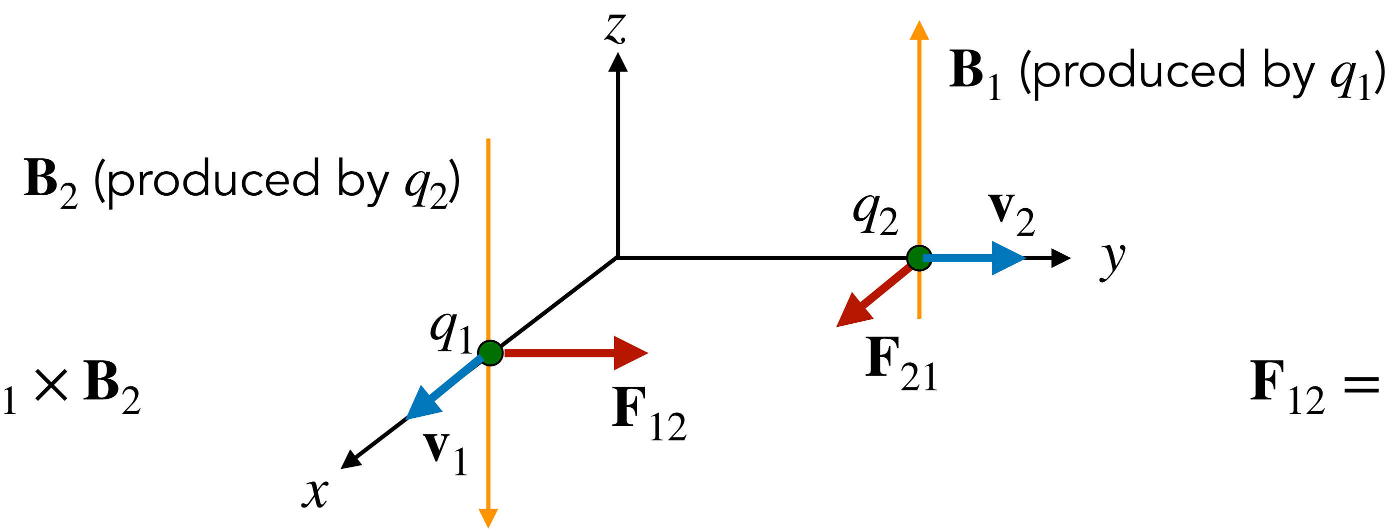


$$F_{21} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}$$

Example Violation of Newton's 3rd Law

Consider two positive charges moving at right angles to each other. Each moving charge creates a magnetic field that the other particle "feels" as a Lorentz force.

$$\mathbf{F}_{21} = q_1 \mathbf{v}_1 \times \mathbf{B}_2$$



$$\mathbf{F}_{12} = q_1 \mathbf{v}_2 \times \mathbf{B}_1$$

- Using right-hand-rules, the Lorentz force on particle 1 (due to the magnetic field produced by particle 2) points in the +y direction.
- Similarly, the Lorentz force on particle 2 (due to the magnetic field produced by particle 2) points in the +x direction.
- Thus, $\mathbf{F}_{12}$ does not point in the opposite direction of $\mathbf{F}_{21}$ and Newton's 3rd law is violated. See Chapter 1 of Taylor for a discussion.